Distribution of Scattering Matrix Elements in Quantum Chaotic Scattering

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Scattering is an important phenomenon which is observed in systems ranging from the micro to macro scale. In the context of nuclear reaction theory the Heidelberg approach was proposed and later demonstrated to be applicable to many chaotic scattering systems. To model the universal properties, stochasticity is introduced to the scattering matrix on the level of the Hamiltonian by using random matrices. A long-standing problem was the computation of the distribution of the off-diagonal scattering-matrix elements. We report here an exact solution to this problem and present analytical results for systems with preserved and with violated time-reversal invariance. Our derivation is based on a new variant of the supersymmetry method. We also validate our results with scattering data obtained from experiments with microwave billiards.

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A large part of our knowledge about quantum systems comes from scattering experiments. Furthermore, even in classical wave systems, observables can often be traced back to a scattering process [1]. Important examples stem from nuclear, atomic and molecular physics, mesoscopic ballistic devices, microwave and elastomechanical billiards [1–4]. Accordingly, the investigation of scattering phenomena has been a subject of major interest from both theoretical and experimental points of view [1–24].

The scattering matrix (S-matrix) is the key quantity in scattering theory. It relates the asymptotic initial and final Hilbert spaces spanned by a quantum system undergoing the scattering process. Owing to the flux conservation requirement the S-matrix is unitary, i.e., $SS^\dagger = S^\dagger S = \mathbb{1}$. The universal features of chaotic scattering systems call for a statistical description. The Heidelberg approach provides a convenient basis for this. The associated S-matrix elements are obtained in terms of the Hamiltonian H describing the scattering center [5], and the coupling vectors W_c which incorporate the information concerning the interaction between the internal states of H and the M open channel states labeled $c = 1, ..., M$. The full system resides in the latter asymptotically before or after the scattering event. The S-matrix elements are obtained as [5, 12]

$$S_{ab}(E) = \delta_{ab} - i2\pi W_a^\dagger G(E) W_b, \tag{1}$$

where the inverse of the propagator $G(E)$ reads

$$G^{-1}(E) = E\mathbb{1}_N - H + i\pi \sum_{c=1}^{M} W_c W_c^\dagger. \tag{2}$$

For the sake of simplicity we restrict the considerations to the case where the average S-matrix is diagonal [25]. This scenario translates to the condition that the coupling vectors W_c are orthogonal, viz., $W_c^\dagger W_d = (\gamma_c/\pi)\delta_{cd}; c, d = 1, ..., M$ [6, 12].

Stochasticity is introduced on the level of the Hamiltonian describing the internal dynamics [5]. In view of the universality conjecture for quantum chaotic systems [26], the Hamiltonian H can be modeled by the Gaussian ensemble of $N \times N$ random matrices with the distribution of H given as [4, 27, 28],

$$\mathcal{P}(H) \propto \exp\Big(-\frac{\beta N}{4v^2} \operatorname{tr} H^2 \Big). \tag{3}$$

Here, v^2 fixes the energy scale and the index β signifies the appropriate symmetry class, i.e., the invariance properties of the Hamiltonian. We focus on the cases $\beta = 1$ (orthogonal ensemble) and $\beta = 2$ (unitary ensemble) which apply to "spinless" systems with, respectively, preserved and violated time-reversal ($\mathcal{T}$) invariance [4, 27, 28].

In their pioneering work [6], using the supersymmetry method [29], Verbaarschot *et al.* calculated the energy correlation function of two S-matrix elements. Further progress in this direction was made in [7] where three- and four-point S-matrix correlation functions were evaluated. While these provide rich information about the scattering process, a full statistical description requires the knowledge of the distributions of the S-matrix elements. The complexity involved in the calculations of such energy correlation functions [6, 7, 30] indicates that the derivation of

the S-matrix distribution functions is an even more challenging task. However, it was partially accomplished in [13] where distribution of the diagonal S-matrix elements was derived. Moreover, in [14] the statistics of transmitted power which is given by $|G_{nm}(E)|^2, n \neq m$, was calculated. These results as well as those for the correlation functions in systems with preserved and violated $\mathcal{T}$ invariance have been verified in microwave experiments [15–22]. A still unsolved problem was the computation of the distribution of the off-diagonal S-matrix elements which could not be tackled with the well-established methods of Refs. [6, 7, 13]. In the present work we provide analytical results for those of their real and imaginary parts. The novelty of our approach lies in that a nonlinear sigma model is constructed based on the *characteristic function* associated with the distributions which is the generating function for the moments. In contrast, the standard supersymmetry approach starts from the generating function for the S-matrix correlations.

We introduce the notation $\wp_s(S_{ab})$, with $s = 1, 2$ to refer to the real and imaginary parts of S_{ab}, respectively. Thus Eq. (1) yields for the off-diagonal ($a \neq b$) elements

$$\wp_s(S_{ab}) = \pi\big((-i)^s W_a^\dagger G W_b + i^s W_b^\dagger G^\dagger W_a\big). \tag{4}$$

Determining distributions for $\wp_s(S_{ab})$, which we denote by $P_s(x_s)$, involves the nontrivial task of performing an ensemble average,

$$P_s(x_s) = \int d[H]\mathcal{P}(H)\delta(x_s - \wp_s(S_{ab})), \quad s = 1, 2. \tag{5}$$

We instead first compute the corresponding characteristic function,

$$R_s(k) = \int d[H]\mathcal{P}(H)\exp(-ik\wp_s(S_{ab})), \tag{6}$$

and then obtain $P_s(x_s)$ as the Fourier transform of $R_s(k)$,

$$P_s(x_s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk R_s(k)\exp(ikx_s). \tag{7}$$

Defining the $2N$-component vector W and the $2N \times 2N$ matrix A_s as

$$W = \begin{bmatrix} W_a \\ W_b \end{bmatrix}, \quad A_s = \begin{bmatrix} 0 & (-i)^s G \\ i^s G^\dagger & 0 \end{bmatrix}, \tag{8}$$

we rewrite $R_s(k)$ as

$$R_s(k) = \int d[H]\mathcal{P}(H)\exp(-ik\pi W^\dagger A_s W). \tag{9}$$

In order to perform the ensemble average, we map the statistical model to superspace. For this we introduce the $2N$-vectors $z^T = [z_a^T, z_b^T]$ and $\zeta^T = [\zeta_a^T, \zeta_b^T]$ consisting of complex commuting and anticommuting (Grassmann) variables, respectively. The supervector is constructed in the usual manner [31] as $\Psi^T = [z^T, \zeta^T]$. With the aid of these vectors, and using the standard multivariate Gaussian integral results for commuting and anticommuting variables, we recast the characteristic function as

$$R_s(k) \; = \; \int d[\Psi]\exp\Big(\frac{i}{2}(\mathbf{W}^\dagger\Psi + \Psi^\dagger\mathbf{W})\Big)\int d[H]\mathcal{P}(H)\exp\Big(\frac{i}{4\pi k}\Psi^\dagger\mathbf{A}_s^{-1}\Psi\Big). \tag{10}$$

Here $\mathbf{A}_s^{-1} = \mathbb{1}_2 \otimes A_s^{-1}$, and $\mathbf{W}^\dagger = [W^\dagger, 0]$ is a $4N$-component vector. Generally, the ensemble average can be done with the help of the supersymmetry method [6, 12, 31], thereby leading to an enormous reduction in the degrees of freedom. However, in the present case the averaging is not straightforward because the matrix $\mathbf{A}_s^{-1}$ is not block diagonal. To resolve this problem we introduce the transformations

$$\begin{gathered} z \to T^+ z,\, z^\dagger \to z^\dagger; \;\; \zeta \to \tfrac{2}{\beta}T^- \zeta,\, \zeta^\dagger \to \zeta^\dagger, \\ T^\pm = \begin{bmatrix} 0 & \pm(-i)^s\mathbb{1}_N \\ -i^s\mathbb{1}_N & 0 \end{bmatrix}. \end{gathered} \tag{11}$$

Here, we have used the fact that the complex quantities z (ζ) and $z^\dagger$ ($\zeta^\dagger$) are independent of each other. The Jacobian of the transformation is $(-1)^N 2^{-2N}$ for $\beta = 1$ and equals $(-1)^N$ for $\beta = 2$. After application of Eq. (11) we have to distinguish between the orthogonal and the unitary cases.

For $\beta = 2$ we obtain

$$R_s(k) = (-1)^N \int d[\Psi] \exp\Big(\frac{i}{2}(\mathbf{U}_s^\dagger \Psi + \Psi^\dagger \mathbf{W})\Big) \int d[H] \mathcal{P}(H) \exp\Big(\frac{i}{4\pi k} \Psi^\dagger \boldsymbol{\mathcal{A}}^{-1} \Psi\Big), \tag{12}$$

with the $4N$-dimensional vector $\mathbf{U}_s^\dagger = \Big[-i^s W_b^\dagger, (-i)^s W_a^\dagger, 0, 0\Big]$ and matrix $\boldsymbol{\mathcal{A}}^{-1} = \mathrm{diag}\left[-(G^{-1})^\dagger, G^{-1}, -(G^{-1})^\dagger, -G^{-1}\right]$, and $\Psi^T = [z^T, \zeta^T]$ as above.

For $\beta = 1$ we decompose the $2N$-vector z into its real and imaginary parts x and y (not to be confused with x_1 and x_2) to construct a $4N$-vector. In addition, we symmetrize the vector ζ using ζ_a^*, ζ_b^* along with ζ_a, ζ_b, thereby also doubling its size. The associated Jacobian equals 2^{2N} and thus cancels that of the transformation Eq. (11). The $8N$-dimensional supervector is given as $\Psi^\dagger = [x_a^T, y_a^T, x_b^T, y_b^T, \zeta_a^\dagger, -\zeta_a^T, \zeta_b^\dagger, -\zeta_b^T]$ and

$$R_s(k) \;=\; (-1)^N \int d[\Psi] \exp\left(i \Psi^\dagger \mathbf{V}_s\right) \int d[H] \mathcal{P}(H) \exp\Big(\frac{i}{4\pi k} \Psi^\dagger \boldsymbol{\mathcal{A}}^{-1} \Psi\Big). \tag{13}$$

In this case $\mathbf{V}_s^T = \left[i^s w_s^-, -i^{s+1} w_s^+, w_s^+, i w_s^-, 0, 0, 0, 0\right]$, where $w_s^\pm = ((-i)^s W_a^T \pm W_b^T)/2$, and $\boldsymbol{\mathcal{A}}^{-1} = \mathrm{diag}[-(G^{-1})^\dagger, G^{-1}, -(G^{-1})^\dagger, -G^{-1}] \otimes \mathbb{1}_2$. Since the matrix $\boldsymbol{\mathcal{A}}^{-1}$ is block diagonal in both Eqs. (12), (13) the ensemble averaging now is straightforward.

Next we employ the standard supersymmetry techniques [31] using the Hubbard-Stratonovitch identity to map the integral over the $8N/\beta$-dimensional vector Ψ to a matrix integral in superspace involving an $8/\beta$-dimensional supermatrix σ of appropriate symmetry. This yields

$$R_s(k) = \int d[\sigma] \exp\Big(-r \,\mathrm{str}\, \sigma^2 - \frac{\beta}{2} \,\mathrm{str}\, \ln \boldsymbol{\Sigma} - \frac{i}{4} \mathbf{F}_s\Big); \tag{14}$$

$$\boldsymbol{\Sigma} = \sigma_E \otimes \mathbb{1}_N + \frac{i}{4k} L \otimes \sum_{c=1}^{M} W_c W_c^\dagger, \quad \sigma_E = \sigma - \frac{E}{4\pi k} \mathbb{1}_{8/\beta},$$

with str denoting the supertrace. Here $r = (4\beta\pi^2 k^2 N)/v^2$, and $L = \mathrm{diag}\,(1, -1, 1, -1) \otimes \mathbb{1}_{2/\beta}$. $\mathbf{F}_s$ equals $\mathbf{V}_s^T \mathbf{L}^{-1/2} \boldsymbol{\Sigma}^{-1} \mathbf{L}^{-1/2} \mathbf{V}_s$ for $\beta = 1$, and $\mathbf{U}_s^\dagger \mathbf{L}^{-1/2} \boldsymbol{\Sigma}^{-1} \mathbf{L}^{-1/2} \mathbf{W}$ for $\beta = 2$ with $\mathbf{L} = L \otimes \mathbb{1}_N$. The supersymmetric representation, Eq. (14), forms one of the key results in this letter.

Using the orthogonality of W_c we eventually obtain

$$\mathrm{str}\, \ln \boldsymbol{\Sigma} = N \,\mathrm{str}\, \ln \sigma_E + \sum_{c=1}^{M} \mathrm{str}\, \ln \Big(\mathbb{1}_{8/\beta} + \frac{i\gamma_c}{4\pi k} \sigma_E^{-1} L\Big),$$

$$\boldsymbol{\Sigma}^{-1} = \sigma_E^{-1} \otimes \mathbb{1}_N - \sigma_E^{-1} \otimes \sum_{c=1}^{M} \frac{\pi}{\gamma_c} W_c W_c^\dagger + \sum_{c=1}^{M} \rho^{(c)} \otimes \frac{\pi}{\gamma_c} W_c W_c^\dagger, \tag{15}$$

with $\rho^{(c)} = (\sigma_E + i\gamma_c/(4\pi k) L)^{-1}$. Furthermore, $\mathbf{F}_s$ equals a linear combination of matrix elements of $\rho^{(c)}$ multiplied with γ_c, where $c = a, b$. Note that in Eq. (15) the first term is of order N while the remaining terms are of order 1. Thus, in order to perform the limit $N \to \infty$, we may apply the saddle point approximation. This leads to a separation of σ into Goldstone modes σ_G and massive modes [31]. The integrals over the latter, being Gaussian ones, can be readily done and yields unity. We are therefore left with an expression involving only the Goldstone modes, and consequently our sigma model reads

$$R_s(k) = \int d\mu(\sigma_G) e^{-\frac{i}{4}\mathbf{F}_s} \prod_{c=1}^{M} \mathrm{sdet}^{-\frac{\beta}{2}} \Big(\mathbb{1}_{8/\beta} + \frac{i\gamma_c}{4\pi k} \sigma_E^{-1} L\Big), \tag{16}$$

with sdet denoting the superdeterminant and σ replaced by σ_G in all the ingredients of Eq. (14). In order to perform the remaining integrals we proceed as in [6, 31] and express σ_G in terms of an $8/\beta$-dimensional supermatrix Q as $\sigma_G = (E/8\pi k)\mathbb{1}_{8/\beta} - (\Delta/8\pi k) Q$ with $Q^2 = -\mathbb{1}_{8/\beta}$. Here, $\Delta = (4v^2 - E^2)^{1/2}$ with $\Delta/(2\pi v^2)$ identified as the celebrated Wigner semicircle. We use the same parametrization of Q as in [6, 12, 32]. For $\beta = 2$, it involves pseudo eigenvalues $\lambda_1 \in (1, \infty), \lambda_2 \in (-1, 1)$, angles $\phi_1, \phi_2 \in (0, 2\pi)$ and four Grassmann variables. For $\beta = 1$ we have three pseudo eigenvalues $\lambda_0 \in (-1, 1), \lambda_1, \lambda_2 \in (1, \infty)$, two O(2) angles $\phi_1, \phi_2 \in (0, 2\pi)$, three SU(2) variables $m, r, s \in (-\infty, \infty)$,

and eight Grassmann variables. The product over the superdeterminants in Eq. (16) involves the pseudo eigenvalues only, and turns out to equal

$$\begin{aligned}\mathcal{F}_{\mathrm{O}} &= \prod_{c=1}^{M} \frac{g_c^+ + \lambda_0}{(g_c^+ + \lambda_1)^{1/2}(g_c^+ + \lambda_2)^{1/2}} \quad \text{for } \beta = 1,\\ \mathcal{F}_{\mathrm{U}} &= \prod_{c=1}^{M} \frac{g_c^+ + \lambda_2}{g_c^+ + \lambda_1} \quad \text{for } \beta = 2.\end{aligned}$$

Here $g_c^{\pm} = (v^2 \pm \gamma_c^2)/(\gamma_c \Delta)$. g_c^+ is related to the transmission coefficient $T_c = 1 - |\overline{S_{cc}}|^2$ as $g_c^+ = 2/T_c - 1$ [6]. The exponential part in Eq. (16) also involves other variables and, in fact, is quite complicated for $\beta = 1$.

For $\beta = 2$ the integrals over the Grassmann variables and over the angles can be performed and we obtain the same distribution for the real and imaginary parts of the S-matrix elements,

$$R_s(k) = 1 - \int_1^{\infty} d\lambda_1 \int_{-1}^{1} d\lambda_2 \frac{k^2}{4(\lambda_1 - \lambda_2)^2}\, \mathcal{F}_{\mathrm{U}}(\lambda_1, \lambda_2)\big(t_a^1 t_b^1 + t_a^2 t_b^2\big) J_0\Big(k\sqrt{t_a^1 t_b^1}\Big), \tag{17}$$

where $J_n(z)$ represents the nth order Bessel function of the first kind, and $t_c^j = \sqrt{|\lambda_j^2 - 1|}/(g_c^+ + \lambda_j)$, $j = 1, 2$. The "1" in Eq. (17) is an Efetov-Wegner contribution [29] which is essential for the correct normalization, $R_s(0) = 1$. The distribution function is obtained using Eq. (7). It is given as

$$\begin{aligned}P_s(x_s) &= \partial^2 f(x_s)/\partial x_s^2,\\ f(x) &= x\Theta(x) + \int_1^{\infty} d\lambda_1 \int_{-1}^{1} d\lambda_2\, \frac{\mathcal{F}_{\mathrm{U}}(\lambda_1, \lambda_2)}{4\pi(\lambda_1 - \lambda_2)^2}\big(t_a^1 t_b^1 + t_a^2 t_b^2\big)\big(t_a^1 t_b^1 - x^2\big)^{-1/2}\Theta(t_a^1 t_b^1 - x^2).\end{aligned} \tag{18}$$

Here, $\Theta(u)$ is the Heaviside function. Since the distributions of the real and the imaginary parts of the S-matrix elements are identical in this case, it is evident that their phases have a uniform distribution and that the joint density of the real and the imaginary parts is a function of $\sqrt{x_1^2 + x_2^2}$ only. The last observation can be used to calculate the distribution of the modulus [33].

For $\beta = 1$, the calculation involved is rather cumbersome. Nevertheless, we managed to perform all but four integrals. For the real part the result reads

$$R_1(k) = 1 + \frac{1}{8\pi} \int_{-1}^{1} d\lambda_0 \int_1^{\infty} d\lambda_1 \int_1^{\infty} d\lambda_2 \int_0^{2\pi} d\psi \mathcal{J}(\lambda_0, \lambda_1, \lambda_2)\mathcal{F}_{\mathrm{O}}(\lambda_0, \lambda_1, \lambda_2) \sum_{n=1}^{4} \kappa_n k^n, \tag{19}$$

where

$$\mathcal{J} = \frac{(1 - \lambda_0^2)|\lambda_1 - \lambda_2|}{2(\lambda_1^2 - 1)^{1/2}(\lambda_2^2 - 1)^{1/2}(\lambda_1 - \lambda_0)^2(\lambda_2 - \lambda_0)^2}.$$

The κ's entering Eq. (19) are functions of

$$\begin{aligned}p_c^j &= \frac{\sqrt{|\lambda_j^2 - 1|}}{8(g_c^+ + \lambda_j)},\ j = 0, 1, 2;\quad p_c^{\pm} = p_c^1 \pm p_c^2,\\ q_c^+ &= \frac{1}{8}\Big(\frac{E}{\Delta} + ig_c^-\Big)\Big(\frac{1}{g_c^+ + \lambda_1} + \frac{1}{g_c^+ + \lambda_2} - \frac{2}{g_c^+ + \lambda_0}\Big),\\ q_c^- &= \frac{1}{8}\Big(\frac{E}{\Delta} + ig_c^-\Big)\Big(\frac{1}{g_c^+ + \lambda_1} - \frac{1}{g_c^+ + \lambda_2}\Big),\end{aligned} \tag{20}$$

and of the complex conjugate of $q_c^{\pm}$, $r_c^{\pm} = (q_c^{\pm})^*$, and the quantities $\omega = 2\sqrt{XY}$, $l = X/Y$, $m = Y/X$, where

$$\begin{aligned}X &= 2p_a^+ + q_a^- e^{-i2\psi} + r_a^- e^{i2\psi},\\ Y &= 2p_b^+ - q_b^- e^{i2\psi} - r_b^- e^{-i2\psi}.\end{aligned}$$

It can be verified that ω^2 is real and takes values from the interval $[0, 1]$ for all values of the parameters involved. The κ's are given as

$$\begin{aligned}&\kappa_1 = \kappa_{11} J_1(k\omega),\quad \kappa_2 = \kappa_{21} J_0(k\omega) + \kappa_{22} J_2(k\omega),\\ &\kappa_3 = \kappa_{31} J_1(k\omega) + \kappa_{32} J_3(k\omega),\\ &\kappa_4 = \kappa_{41} J_0(k\omega) + \kappa_{42} J_2(k\omega) + \kappa_{43} J_4(k\omega),\end{aligned}$$

with the entries κ_{ij}, $i,j=1,\cdots,4$ given in the appendix.

The characteristic function for the imaginary part, $R_2(k)$, is obtained by multiplying $-i$ to the right hand side of the expressions for $q_c^{\pm}$ in Eq. (20), and changing $r_c^{\pm}$ accordingly. The distributions of the real and the imaginary parts of the S-matrix elements can be cast into the form,

$$P_s(x_s) = \delta(x_s) + \frac{\partial f_1}{\partial x_s} + \frac{\partial^2 f_2}{\partial x_s^2} + \frac{\partial^3 f_3}{\partial x_s^3} + \frac{\partial^4 f_4}{\partial x_s^4}; \tag{21}$$

$$\begin{aligned}
f_1 &= \langle \kappa_{11} x_s/\omega \rangle\,, \\
f_2 &= -\left\langle \kappa_{21} + \kappa_{22}\big(1-2x_s^2/\omega^2\big)\right\rangle, \\
f_3 &= -\left\langle \big[\kappa_{31} + \kappa_{32}\big(3-4x_s^2/\omega^2\big)\big] x_s/\omega \right\rangle, \\
f_4 &= \left\langle \big[\kappa_{41} + \kappa_{42}\big(1-2x_s^2/\omega^2\big) + \kappa_{43}\big(1-8x_s^2/\omega^2+8x_s^4/\omega^4\big)\big] \right\rangle.
\end{aligned}$$

Here the angular brackets represent the following:

$$\langle h \rangle = \frac{1}{16\pi^2}\int_{-1}^{1} d\lambda_0 \int_1^{\infty} d\lambda_1 \int_1^{\infty} d\lambda_2 \int_0^{2\pi} d\psi\, \mathcal{J}(\lambda_0,\lambda_1,\lambda_2)\mathcal{F}_{\mathrm{O}}(\lambda_0,\lambda_1,\lambda_2)\, 2h(\omega^2-x_s^2)^{-1/2}\Theta(\omega^2-x_s^2).$$

Details of the supersymmetry calculations and further results will be given elsewhere [33].

We evaluated Eqs. (17) and (18) numerically using MATHEMATICA [34]. In Fig. 1 we compare for $\beta=2$ the analytical results for characteristic functions and distributions with simulations obtained with an ensemble of 50000 random matrices H of dimensions 200×200 from the GUE. The agreement is excellent. Unfortunately, there were no experimental data available for this case because complete violation of the $\mathcal{T}$ invariance could not be achieved. For $\beta=1$ we found that $R_s(k)$ is best evaluated using the Efetov variables $\theta_0,\theta_1,\theta_2$ $(0<\theta_0<\pi, 0<\theta_{1,2}<\infty)$ [29]. These are related to the λ's as $\lambda_0=\cos\theta_0$ and $\lambda_{1,2}=\cosh(\theta_1\pm\theta_2)$. The numerical evaluation of the fourth derivative needed for the computation of the distributions $P_s(x_s)$ is not feasible. We therefore instead determined them with the help of Eq. (7), considering a cut-off for k. This approach works well for a sufficiently flat distribution, whereas, if it is highly localized, it is advantageous to consider the corresponding characteristic function instead.

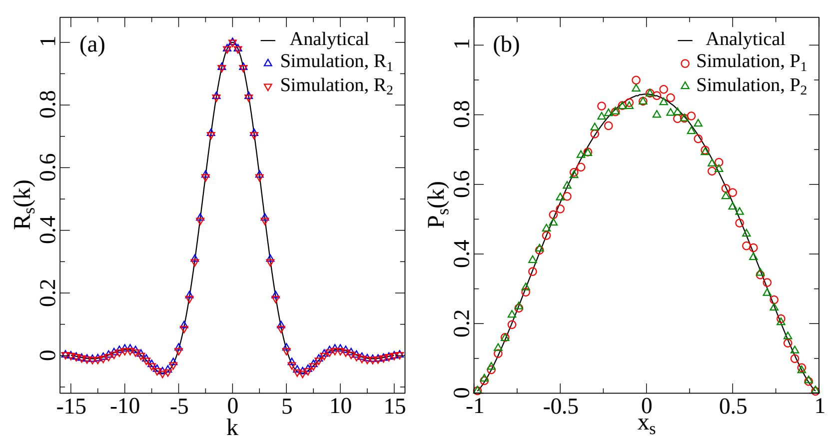

FIG. 1: Comparison of analytical and simulation results for $\beta=2$: (a) Characteristic functions (b) Distributions of the real and the imaginary parts of S_{ab} for the choice of parameters $M=3, E=0.25, v=1, \gamma_1=0.8, \gamma_2=1.0, \gamma_3=1.2, a=1, b=2$.

We tested our analytical results for $\beta=1$ with experimental data. The latter stem from microwave measurements. To realize a chaotic scattering system, a microwave billiard with the shape of a tilted-stadium [20–22] was chosen and the resonator modes were coupled to the exterior via two antennas attached to it. An ensemble of several chaotic systems was obtained by introducing a small scatterer into the microwave billiard and moving it to six different positions [36]. For the determination of the off-diagonal S-matrix elements a vector network analyzer (VNA) coupled microwave power into the resonator via one antenna and determined magnitude and phase of the transmitted signal at the other one. The frequency range was chosen such that only the vertical component of the electric field strength was excited. Then the Helmholtz equation is mathematically equivalent to the Schrödinger equation of a quantum tilted stadium billiard. The dynamics of the corresponding classical billiard is chaotic [35]. The S-matrix elements

were measured in steps of 100 kHz in a range from 1-25 GHz. Disturbing effects due to the coaxial cables connecting the VNA to the microwave billiard were eliminated by calibration. Furthermore, the fluctuation properties of the S-matrix elements were evaluated in frequency windows of 1 GHz in order to guarantee a negligible secular variation of the dissipation in the walls and of the coupling between the modes in the coaxial cables and the resonator modes, which are incorporated in the coupling vectors W_c in Eq. (2). More details concerning the experimental setup and the measurements are provided in [20, 21]. In Figs. 2 and 3, we test the analytical results with experimental data for the frequency ranges 10-11 GHz and 24-25 GHz, corresponding to a ratio of the average resonance width Γ and average resonance spacing d, $\Gamma/d = 0.234$ and, respectively, $\Gamma/d = 1.21$. The agreement is very good.

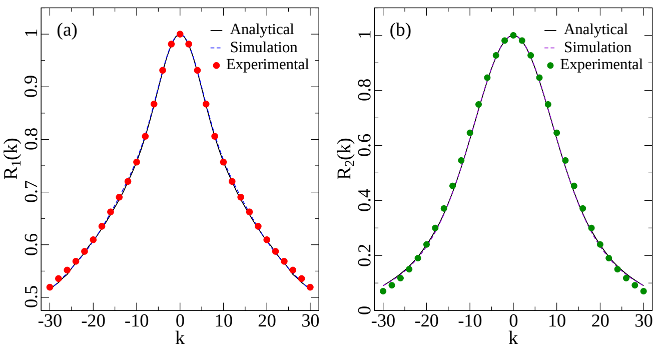

FIG. 2: Characteristic functions R_1 and R_2 corresponding to the real and the imaginary parts of S_{12} for $\beta = 1$. Comparison between the analytical results and the microwave experiment data for the frequency range 10-11 GHz [20–22].

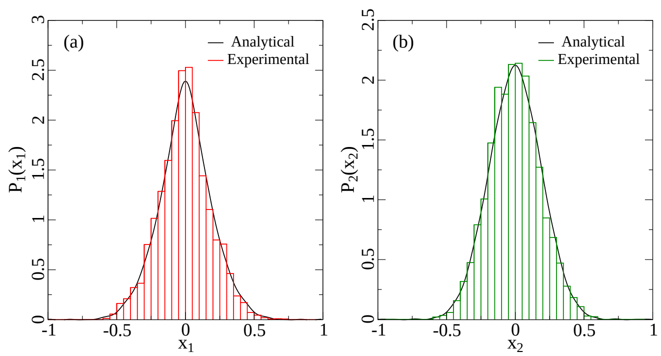

FIG. 3: Distributions of the real (P_1) and the imaginary (P_2) parts of S_{12} for $\beta = 1$. Comparison between analytical results and microwave experiment data for the frequency range 24-25 GHz [20–22].

To conclude, we solved the long-standing problem of deriving the full distribution of the off-diagonal S-matrix elements for arbitrary values of the parameters involved. We accomplished this task by introducing a novel route to the sigma model based on the characteristic function. We verified our analytical results with numerical simulations and found excellent agreements. Furthermore, we tested our results with experimental data obtained with microwave billiards, and thus presented a new confirmation of the random matrix universality conjecture.

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APPENDIX: EXPLICIT EXPRESSIONS FOR THE κ_{ij}

$$\kappa_{11} = -(9/8)\{p_a^+ m^{1/2}\}_+,$$

$$\kappa_{21} = -(1/4)(128 p_a^0 p_b^0 + 14 p_a^+ p_b^+ + 32 p_a^- p_b^-) + \{3e^{i2\psi}(p_a^- q_b^+ - p_b^- r_a^+)\}_- + \{e^{-4i\psi} q_a^- r_b^-\}_+,$$

$$\kappa_{22} = -(1/4)\{(p_a^+ p_a^+ - 4 q_a^- r_a^-)m\}_+,$$

$$\begin{aligned}
\kappa_{31} = &\big\{2\big[(p_a^+ p_a^+ + q_a^- r_a^-)m^{1/2} + 2(8p_a^0 p_b^0 + p_a^+ p_b^+ + p_a^- p_b^-)l^{1/2}\big](e^{i2\psi} q_b^- + e^{-i2\psi} r_b^-)\big\}_- \\
&+\big\{2\big[(p_a^+ p_b^- + 4 p_a^- p_b^+)m^{1/2} + p_b^+ p_b^- l^{1/2}\big](e^{-i2\psi} q_a^+ + e^{i2\psi} r_a^+)\big\}_- + \big\{\big[16 p_a^0(2p_a^0 p_b^+ - 3p_b^0 p_a^+) \\
&-6p_a^+(q_a^+ q_b^+ + r_a^+ r_b^+) + 2p_b^+(4q_a^+ r_a^+ - q_a^- r_a^-) - 4p_a^-(p_a^+ p_b^- - 2p_a^- p_b^+) + 3p_a^+(q_a^- q_b^- + r_a^- r_b^- - p_a^+ p_b^+) \\
&+(e^{-i4\psi}/2) q_a^-(4p_a^+ r_b^- - 3p_b^+ q_a^-) + (e^{i4\psi}/2) r_a^-(4p_a^+ q_b^- - 3p_b^+ r_a^-)\big]m^{1/2}\big\}_+ \\
&+\big\{\big[(e^{-i4\psi}/2) q_a^-(2e^{-i2\psi} q_a^- r_b^- - 8e^{i2\psi} r_a^+ r_b^+) + (e^{i4\psi}/2) r_a^-(2e^{i2\psi} q_b^- r_a^- - 8e^{-i2\psi} q_a^+ q_b^+)\big]m^{1/2}\big\}_-,
\end{aligned}$$

$$\kappa_{32} = \big\{p_a^+\big[(p_a^+ p_a^+ + 2q_a^- r_a^-) + (3/2)(e^{-i4\psi} q_a^- q_a^- + e^{i4\psi} r_a^- r_a^-)\big]m^{3/2}\big\}_+ + \big\{(2p_a^+ p_a^+ + q_a^- r_a^-)(e^{-i2\psi} q_a^- + e^{i2\psi} r_a^-)m^{3/2}\big\}_-,$$

$$\begin{aligned}
\kappa_{41} \;=\; & 32\big[2p_a^0 p_a^0(p_b^- - e^{i2\psi} q_b^+)(p_b^- - e^{-i2\psi} r_b^+) + 2p_b^0 p_b^0(p_a^- + e^{-i2\psi} q_a^+)(p_a^- + e^{i2\psi} r_a^+) \\
+\; & p_a^0 p_b^0\big((p_a^+ + e^{-i2\psi} q_a^-)(p_b^+ - e^{-i2\psi} r_b^-) + (p_a^+ + e^{i2\psi} r_a^-)(p_b^+ - e^{i2\psi} q_b^-)\big)\big] \\
+\; & 256 p_a^0 p_a^0 p_b^0 p_b^0 + (p_a^+ + e^{-i2\psi} q_a^-)^2(p_b^+ - e^{-i2\psi} r_b^-)^2 + (p_a^+ + e^{i2\psi} r_a^-)^2(p_b^+ - e^{i2\psi} q_b^-)^2 \\
+\; & 4\big[(p_a^+ + e^{-i2\psi} q_a^-)(p_b^+ - e^{i2\psi} q_b^-) - 2(p_a^- + e^{-i2\psi} q_a^+)(p_b^- - e^{i2\psi} q_b^+)\big] \\
& \times\big[(p_a^+ + e^{i2\psi} r_a^-)(p_b^+ - e^{-i2\psi} r_b^-) - 2(p_a^- + e^{i2\psi} r_a^+)(p_b^- - e^{-i2\psi} r_b^+)\big],
\end{aligned}$$

$$\begin{aligned}
\kappa_{42} = &-32 p_a^0 p_b^0\big[(p_a^+ + e^{-i2\psi} q_a^-)(p_a^+ + e^{i2\psi} r_a^-)m + (p_b^+ - e^{i2\psi} q_b^-)(p_b^+ - e^{-i2\psi} r_b^-)l\big] \\
&-2\big[(p_a^+ + e^{-i2\psi} q_a^-)(p_b^+ - e^{i2\psi} q_b^-) - 2(p_a^- + e^{-i2\psi} q_a^+)(p_b^- - e^{i2\psi} q_b^+)\big]\big[(p_a^+ + e^{i2\psi} r_a^-)^2 m + (p_b^+ - e^{-i2\psi} r_b^-)^2 l\big] \\
&-2\big[(p_a^+ + e^{i2\psi} r_a^-)(p_b^+ - e^{-i2\psi} r_b^-) - 2(p_a^- + e^{i2\psi} r_a^+)(p_b^- - e^{-i2\psi} r_b^+)\big]\big[(p_a^+ + e^{-i2\psi} q_a^-)^2 m + (p_b^+ - e^{i2\psi} q_b^-)^2 l\big],
\end{aligned}$$

$$\kappa_{43} = (p_a^+ + e^{-i2\psi} q_a^-)^2(p_a^+ + e^{i2\psi} r_a^-)^2 m^2 + (p_b^+ - e^{i2\psi} q_b^-)^2(p_b^+ - e^{-i2\psi} r_b^-)^2 l^2.$$

In the above equations, we introduced the notation $\{\mathcal{E}(a,b,l,m,\psi)\}_\pm := \mathcal{E}(a,b,l,m,\psi) \pm \mathcal{E}(b,a,m,l,-\psi)$ with $\mathcal{E}$ an expression involving a, b, l, m, ψ.

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